Physics 604
Problem Set 5
Due Nov. 18, 2010

1) a) First we need to find the appropriate expressions for the charge density. The full volume integral must yield $\pm q$ for each charge. Therefore

$$\rho(r,\theta,\phi)=\frac{\delta(r-a)}{r^2}\delta(\cos\theta)\left[\delta(\phi)+\delta(\phi-\pi/2)-\delta(\phi-\pi)-\delta(\phi-3\pi/2)\right]$$
$$q_{lm}=qa^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)\left[1+e^{-im\pi/2}-e^{-im\pi}-e^{-3im\pi/2}\right]$$
$$q_{lm}=qa^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\begin{Bmatrix}(-1)^{(l+m)/2}\dfrac{(l+m-1)!!}{(l-m)!!} & l+m\text{ even}\\ 0 & l+m\text{ odd}\end{Bmatrix}\left[1+(-i)^m\right]\left[1-(-1)^m\right]$$

by a problem in *Arfken* (3rd edition problem 12.5.3). Clearly m, and by implication also l, must be odd, but otherwise, they are unrestricted except for the usual conditions. Note: Arfken's sign convention on $P^m_{l,Arfken}(x)=(-1)^m P^m_{l,Jackson}(x)$; Jackson agrees with Abramowitz and Stegun, the famous *AMS-55.* They all agree on the sign convention for Y_{lm}, which derives from the *Condon and Shortley* phase conventions.

b) By performing the normalization integrals properly one obtains

$$\rho(r,\theta,\phi)=\frac{q\delta(r-a)}{2\pi r^2}\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right]-\frac{2q\delta(r)}{4\pi r^2}.$$

This means

$$q_{lm}=\int r'^2dr'd\Omega' r'^l Y^*_{lm}(\theta',\phi')\rho(r',\theta',\phi')$$
$$q_{00}=\frac{-2q}{\sqrt{4\pi}}+\frac{q}{\sqrt{4\pi}}+\frac{q}{\sqrt{4\pi}}=0$$
$$q_{10}=0+qa\sqrt{\frac{3}{4\pi}}-qa\sqrt{\frac{3}{4\pi}}=0$$
$$q_{20}=0+qa^2\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}-\frac{1}{2}\right)+qa^2\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}-\frac{1}{2}\right)=2q^2\sqrt{\frac{5}{4\pi}}$$
$$q_{l0}=0+qa^l\sqrt{\frac{2l+1}{4\pi}}P_l(1)+qa^l\sqrt{\frac{2l+1}{4\pi}}P_l(-1)=qa^l\sqrt{\frac{2l+1}{4\pi}}\left[1+(-1)^l\right].$$

So the azimuthally symmetric moments are non-zero only for even l. Because the distribution is azimuthally symmetrical, only $m=0$ moments are non-zero by orthogonality of $e^{-im\phi'}$ with the constant function in ϕ'.

c) Clearly

$$\Phi(\vec{x})=\frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi q_{lm}}{(2l+1)r^{l+1}}Y_{lm}(\theta,\phi)=\frac{1}{4\pi\varepsilon_0}\sum_{\substack{l=2\\ even}}^{\infty}\frac{2qa^l}{r^{l+1}}P_l(\cos\theta).$$

The dominant term at large radii is

$$\Phi(\vec{x})\approx\frac{1}{4\pi\varepsilon_0}\frac{2qa^2}{r^3}P_2(\cos\theta).$$

The expansion in the $x-y$ plane is

$$\Phi(\vec{x})=\frac{2q}{4\pi\varepsilon_0 r}\sum_{\substack{l=2\\ even}}^{\infty}\frac{a^l}{r^l}P_l(0)=\frac{2q}{4\pi\varepsilon_0 r}\sum_{\substack{l=2\\ even}}^{\infty}(-1)^{l/2}\frac{a^l}{r^l}\frac{(l-1)!!}{l!!}.$$

d) Calculating with Coulomb's Law in the $x-y$ plane

$$\Phi(\vec{x})=\frac{2q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+a^2}}-\frac{1}{r}\right]=\frac{2q}{4\pi\varepsilon_0}\left[\frac{1}{r}\left(1+a^2/r^2\right)^{-1/2}-\frac{1}{r}\right],$$

which expands just as in c)

$$\Phi(\vec{x})=\frac{2q}{4\pi\varepsilon_0 r}\left[-\frac{1}{2}\frac{a^2}{r^2}+\frac{1\cdot 3}{2\cdot 4}\frac{a^4}{r^4}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{a^6}{r^6}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\frac{a^8}{r^8}+\cdots\right].$$

2) a) The distribution is azimuthally symmetrical and so only $m=0$ contributes. Next notice that $\sin^2\theta=1-\cos^2\theta=1-\sqrt{\frac{4\pi}{5}}\frac{2}{3}Y_{20}-\frac{1}{3}=\frac{2}{3}\sqrt{4\pi}Y_{00}-\sqrt{\frac{4\pi}{5}}\frac{2}{3}Y_{20}$, and so

$$q_{00}=\frac{1}{64\pi}\int_0^{\infty}r^4e^{-r}dr\left(\frac{2}{3}\sqrt{4\pi}\right)=\frac{4!}{32\cdot 3\pi}=\frac{1}{4\pi}\sqrt{4\pi}$$

$$q_{20}=\frac{1}{64\pi}\int_0^{\infty}r^6e^{-r}dr\left(-\frac{2}{3}\sqrt{\frac{4\pi}{5}}\right)=-\frac{6!}{32\cdot 3\pi}=-\frac{30}{4\pi}\sqrt{\frac{4\pi}{5}}$$

$$\Phi(\vec{x})=\frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi q_{lm}}{(2l+1)r^{l+1}}Y_{lm}(\theta,\phi)=\frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}-\frac{6}{r^3}P_2(\cos\theta)\right]$$

The expressions are actually in scaled units, as discussed in part c).

b) In order to compute the total potential we'll need a variety of indefinite integrals. You can look them up or follow this table

$$\int e^{-x}dx = -e^{-x}$$
$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x}$$
$$\int x^2e^{-x}dx = -x^2e^{-x} + 2\int xe^{-x}dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x}$$
$$\int x^3e^{-x}dx = -x^3e^{-x} + 3\int x^2e^{-x}dx = -x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x}$$
$$\int x^4e^{-x}dx = -x^4e^{-x} + 4\int x^3e^{-x}dx = -x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} - 24e^{-x}$$
$$\int x^5e^{-x}dx = -x^5e^{-x} + 5\int x^3e^{-x}dx =$$
$$-x^5e^{-x} - 5x^4e^{-x} - 20x^3e^{-x} - 60x^2e^{-x} - 120xe^{-x} - 120e^{-x}$$
$$\int x^6e^{-x}dx = -x^6e^{-x} + 6\int x^3e^{-x}dx =$$
$$-x^5e^{-x} - 6x^5e^{-x} - 30x^4e^{-x} - 180x^3e^{-x} - 360x^2e^{-x} - 720xe^{-x} - 720e^{-x}$$

The total potential is

$$\begin{aligned}\Phi(\vec{x}) &= \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3\vec{x}' \\ &= \frac{1}{4\pi\varepsilon_0}\int\rho(\vec{x}')\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{(2l+1)}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')d^3\vec{x}'Y_{lm}(\theta,\phi) \\ &= \frac{1}{4\pi\varepsilon_0}4\pi\left[\frac{1}{r}\int_0^r r'^4e^{-r'}dr' + \int_r^{\infty}r'^3e^{-r'}dr'\right]k00Y_{00}(\theta,\phi) \\ &-\frac{1}{4\pi\varepsilon_0}\frac{4\pi}{5}\left[\frac{1}{r^3}\int_0^r r'^6e^{-r'}dr' + r^2\int_r^{\infty}r'e^{-r'}dr'\right]k20Y_{20}(\theta,\phi)\end{aligned}$$

where the division in the expression for the expansion coefficient accounts for the normalization of the orthogonal set. Therefore

$$\delta(\phi-\phi') = \sum_{m=1}^{\infty}\frac{2}{\beta}\sin\frac{m\pi\phi'}{\beta}\sin\frac{m\pi\phi}{\beta}.$$

3) a) The potential in each of the three regions will be called $\Phi_I(r,\theta)$, $\Phi_{II}(r,\theta)$, and $\Phi_{III}(r,\theta)$. By the mid-term, for cylindrical geometry, for example

$$\Phi_I(r,\theta) = \sum_{\nu=0}^{\infty}\left(A_\nu r^\nu + B_\nu r^{-\nu}\right)\cos\nu\theta$$

The coefficients for region II will be C_ν, D_ν and region III will be E_ν (there can be no singularity at the origin. The $\sin\theta$ terms will be zero as we're matching to $-E_0\cos\theta$ at large radii, the cylindrically symmetrical parts are zero because there is no net *free* charge. In the same manner as the mid-term, only $\nu = \pm 1$ are non-zero. The boundary conditions at $r = a$ yield

$$\varepsilon_0 E_1 = \varepsilon\left(C_1 - D_1 / a^2\right)$$
$$E_1 = C_1 + D_1 / a^2$$

The boundary conditions at $r = b$ yield

$$\varepsilon_0\left(A_1 - B_1 / b^2\right) = \varepsilon\left(C_1 - D_1 / b^2\right) = \varepsilon_0\left(-E_0 - B_1 / b^2\right)$$
$$A_1 + B_1 / b^2 = C_1 + D_1 / b^2 = -E_0 + B_1 / b^2$$

Now algebra yields

$$A_1 = -E_0$$
$$\frac{B_1}{b^2} = \frac{\left(b^2 - a^2\right)K}{K^2 a^2 - b^2}\left(-E_0\right) = E_0 \frac{\left(b^2 - a^2\right)\left(1 - \varepsilon_0^2 / \varepsilon^2\right)}{\left(1 + \varepsilon_0 / \varepsilon\right)^2 b^2 - \left(1 - \varepsilon_0 / \varepsilon\right)^2 a^2}$$
$$E_1 = \frac{K^2 b^2 - b^2}{K^2 a^2 - b^2}\left(-E_0\right) = -4E_0 \frac{\left(\varepsilon_0 / \varepsilon\right) b^2}{\left(1 + \varepsilon_0 / \varepsilon\right)^2 b^2 - \left(1 - \varepsilon_0 / \varepsilon\right)^2 a^2}$$
$$C_1 = \frac{\left(1 + \varepsilon_0 / \varepsilon\right)}{2} \frac{K^2 b^2 - b^2}{K^2 a^2 - b^2}\left(-E_0\right) = -2E_0 \frac{\left(1 + \varepsilon_0 / \varepsilon\right)\left(\varepsilon_0 / \varepsilon\right) b^2}{\left(1 + \varepsilon_0 / \varepsilon\right)^2 b^2 - \left(1 - \varepsilon_0 / \varepsilon\right)^2 a^2}$$
$$\frac{D_1}{a^2} = \frac{\left(1 - \varepsilon_0 / \varepsilon\right)}{2} \frac{K^2 b^2 - b^2}{K^2 a^2 - b^2}\left(-E_0\right) = -2E_0 \frac{\left(1 - \varepsilon_0 / \varepsilon\right)\left(\varepsilon_0 / \varepsilon\right) b^2}{\left(1 + \varepsilon_0 / \varepsilon\right)^2 b^2 - \left(1 - \varepsilon_0 / \varepsilon\right)^2 a^2}$$
$$K^2 = \frac{\left(1 - \varepsilon_0 / \varepsilon\right)^2}{\left(1 + \varepsilon_0 / \varepsilon\right)^2}$$

c) In the limit $a \to 0$ the coefficients become

$$A_1 = -E_0$$
$$\frac{B_1}{b^2} = -K\left(-E_0\right)$$
$$C_1 = \frac{\left(1 + \varepsilon_0 / \varepsilon\right)}{2}\left(1 - K^2\right)\left(-E_0\right) = -2E_0 \frac{\varepsilon_0 / \varepsilon}{\left(1 + \varepsilon_0 / \varepsilon\right)}$$
$$D_1 = 0$$

and the potential becomes

$$\Phi(r,\theta)=\begin{cases}-E_0 r\cos\theta+E_0 K\dfrac{b^2}{r}\cos\theta & r>b\\ -E_0\dfrac{2\varepsilon_0/\varepsilon}{1+\varepsilon_0/\varepsilon}r\cos\theta & r<b\end{cases}$$

To get the other solution, take the limit $b\to\infty$. Then

$$E_1=-4E_0\frac{(\varepsilon_0/\varepsilon)}{(1+\varepsilon_0/\varepsilon)^2}$$

$$C_1=-2E_0\frac{(\varepsilon_0/\varepsilon)}{(1+\varepsilon_0/\varepsilon)}$$

$$\frac{D_1}{a^2}=-2E_0\frac{(1-\varepsilon_0/\varepsilon)(\varepsilon_0/\varepsilon)}{(1+\varepsilon_0/\varepsilon)^2}$$

Regrouping with

$$\hat{E}_0=2E_0\frac{\varepsilon_0/\varepsilon}{1+\varepsilon_0/\varepsilon},$$

the asymptotic field at large radii, one obtains

$$\Phi(r,\theta)=\begin{cases}-\hat{E}_0 r\cos\theta-\hat{E}_0 K\dfrac{a^2}{r}\cos\theta & r>a\\ -\hat{E}_0\dfrac{2}{1+\varepsilon_0/\varepsilon}r\cos\theta & r<a\end{cases}$$

or

$$\Phi(r,\theta)=\begin{cases}-\hat{E}_0 r\cos\theta+\hat{E}_0\dfrac{1-\varepsilon/\varepsilon_0}{1+\varepsilon/\varepsilon_0}\dfrac{a^2}{r}\cos\theta & r>a\\ -\hat{E}_0\dfrac{2(\varepsilon/\varepsilon_0)}{1+\varepsilon/\varepsilon_0}r\cos\theta & r<a\end{cases}$$

This result is the same as the first part of this section with the roles of ε_0 and ε swapped and

$b\to a$.

4) a) The main "trick" to this problem is to realize that if the electric field is totally radial, and the same in the two regions, then the boundary conditions at the dielectric boundary are automatically satisfied: the tangential $\vec{E}$ field is continuous by assumption, and the normal

$\vec{D} = \varepsilon\vec{E}$ is continuous automatically because it is zero on either side of the dielectric boundary. Since all the boundary conditions are satisfied by a purely radial field, by the uniqueness theorem, this must be THE solution. By Gauss's Law applied to the displacement vector

$$\int \vec{D}\cdot\vec{n}da = D_{1,r}2\pi r^2 + D_{2,r}2\pi r^2 = Q$$
$$(\varepsilon_0 + \varepsilon)E_r 2\pi r^2 = Q$$
$$\therefore E_r = \frac{Q}{(\varepsilon_0+\varepsilon)2\pi r^2}$$

One could also use the general expansion for the potential in the region between the spheres, assuming the *z*-axis is along the rotational symmetry direction of the problem. The potential in the region between the spheres is in general

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-l-1}\right)P_l(\cos\theta),$$

where possible "leakage" of the field into or out of the dielectric could be handled by having non-zero higher l values. Fortunately, the boundary conditions at $r = a$ and $r = b$ exclude such a possibility! For example

$$\Phi(a,\theta) = cons = \sum_{l=0}^{\infty}\left(A_l a^l + B_l a^{-l-1}\right)P_l(\cos\theta)$$
$$\rightarrow A_l a^l + B_l a^{-l-1} = 0 \qquad l \neq 0$$

by orthogonality of P_l and the constant function. Likewise,

$$A_l b^l + B_l b^{-l-1} = 0 \qquad l \neq 0$$

These two conditions imply that $A_l = B_l = 0$ for $l \neq 0$. Once one knows $\Phi(r,\theta) = A_0 + B_0 / r$, it is a short calculation to find the electric field as above.

b) $\vec{D} = \vec{E} = 0$ inside the surface of the conducting shells. Therefore the "free" surface charge density on the surface is given simply by the value of the displacement on the surface.

$$\sigma_{free} = \vec{D}\cdot\vec{n} = \varepsilon_i E_r(r=a) = \frac{\varepsilon_i}{\varepsilon_0+\varepsilon}\frac{Q}{2\pi a^2}.$$

On the surface of the sphere exposed to vacuum, the charge density is

$$\sigma = \frac{\varepsilon_0}{\varepsilon_0+\varepsilon}\frac{Q}{2\pi a^2}.$$

On the surface of the inner sphere exposed to the dielectric, the charge density is

$$\sigma = \frac{\varepsilon}{\varepsilon_0 + \varepsilon} \frac{Q}{2\pi a^2}.$$

c) The polarization surface charge is

$$\begin{aligned}\sigma_{pol} &= -\vec{P} \cdot \vec{n} = -\left(\varepsilon - \varepsilon_0\right) E_r \\ &= \frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 + \varepsilon} \frac{Q}{2\pi a^2}\end{aligned}.$$

The “effective” charge density on the dielectric side is therefore

$$\sigma_{effective} = \left[\frac{\varepsilon}{\varepsilon_0 + \varepsilon} + \frac{\varepsilon_0 - \varepsilon}{\varepsilon_0 + \varepsilon} \right] \frac{Q}{2\pi a^2} = \frac{\varepsilon_0}{\varepsilon_0 + \varepsilon} \frac{Q}{2\pi a^2},$$

just as on the vacuum surface. This charge yields the proper total electric field in the radial direction.